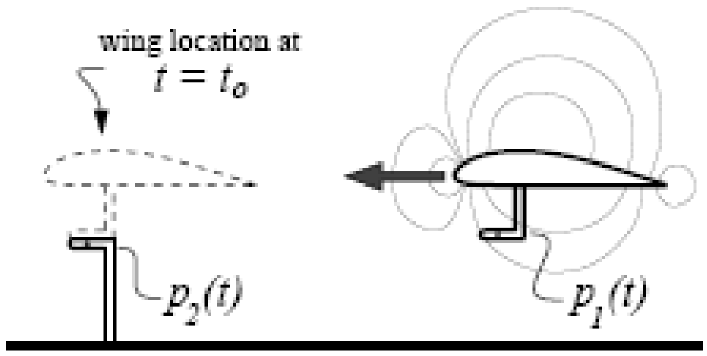
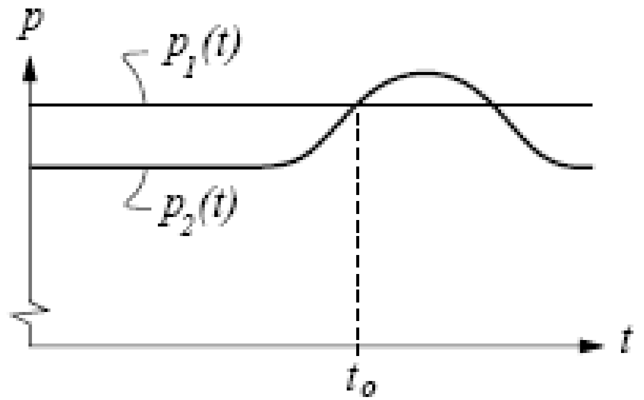
< 2.9. Substantial derivative >

❖ Sensed rates of change

- The rate of change reported by a flow sensor clearly depends on the motion of the sensor. For example, the pressure reported by a static-pressure sensor mounted on an airplane in level flight shows zero rate of change. But a ground pressure sensor reports a nonzero rate as the airplane rapidly flies by a few meters overhead.



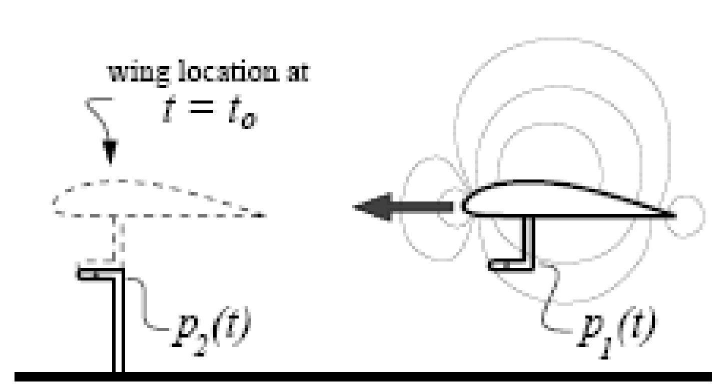


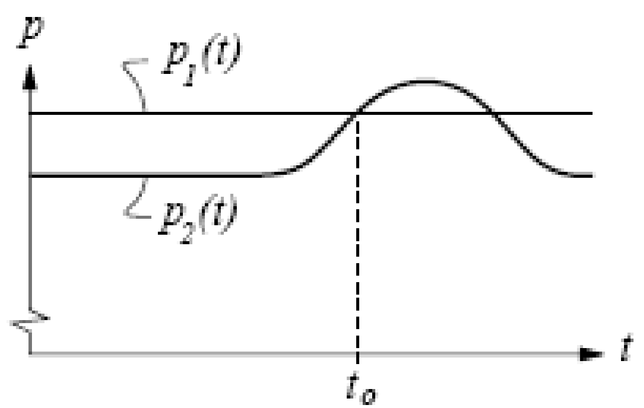
< 2.9. Substantial derivative >

❖ Sensed rates of change

- Note that although the two sensors measure the same instantaneous static pressure at the same point (at time $t=t_0$), the measured <u>*time rates*</u> are different.

$$p_1(t_0) = p_2(t_0) \qquad but \qquad \frac{dp_1}{dt}(t_0) \neq \frac{dp_2}{dt}(t_0)$$



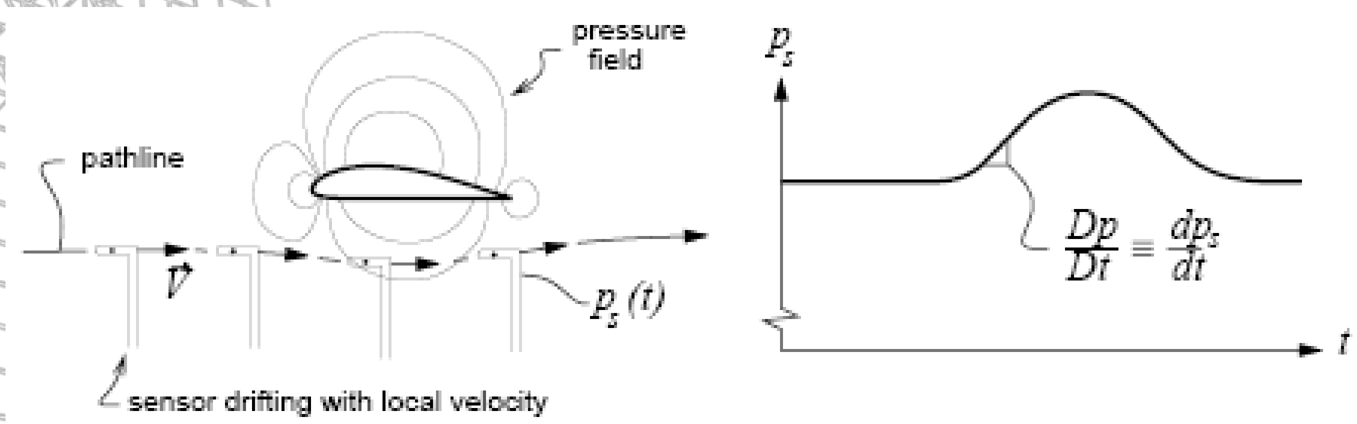


< 2.9. Substantial derivative >

Drifting sensor

- We will now imagine a sensor *drifting with a fluid element*. In effect, the sensor follows the element's path-line coordinates $x_s(t), y_s(t), z_s(t)$, whose time rates of change are just the local flow velocity components

$$\frac{dx_s}{dt} = u(x_s, y_s, z_s, t) \ , \ \frac{dy_s}{dt} = u(x_s, y_s, z_s, t) \ , \ \frac{dz_s}{dt} = u(x_s, y_s, z_s, t)$$



< 2.9. Substantial derivative >

❖ Drifting sensor

- Consider a flow field quantity to be observed by the drifting sensor, such as the static pressure $p(x,y,z,t)$.As the sensor moves through this field, the instantaneous pressure value reported by the sensor is then simply

$$p_s(t) = p\left(x_s(t), y_s(t), z_s(t), t\right)$$

- This $p_s(t)$ signal is similar to $p_2(t)$ in the example above, but not quite the same, since the p_2 sensor moves in a straight line relative to the wing rather than following a path-line like the p_s sensor.

< 2.9. Substantial derivative >

❖ Substantial derivative definition

- The time rate of change of $p_s(t)$ can be computed from above equation using the chain rule.

$$\frac{dp_s}{dt} = \frac{\partial p}{\partial x}\frac{\partial x_s}{\partial t} + \frac{\partial p}{\partial y}\frac{\partial y_s}{\partial t} + \frac{\partial p}{\partial z}\frac{\partial z_s}{\partial t} + \frac{\partial p}{\partial t}$$

< 2.9. Substantial derivative >

❖ Substantial derivative definition

- But since dx_s/dt etc. are simply the local fluid velocity components, this rate can be expressed using the flow-field properties alone.

$$\frac{dp_s}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} \equiv \frac{Dp}{Dt}$$

- The middle expression, conveniently denoted as *Dp/Dt* in shorthand, is called the *substantial derivative* of *p*.

< 2.9. Substantial derivative >

❖ Substantial derivative definition

- Although we used the pressure in this example, the substantial derivative can be computed for *any* flow-field quantity (density, temperature, even velocity)which is a function of *x,y,z,t*.

$$\frac{D(\ )}{dt}=\frac{\partial(\ )}{\partial t}+u\frac{\partial(\ )}{\partial x}+v\frac{\partial(\ )}{\partial y}+w\frac{\partial(\ )}{\partial z}=\frac{\partial(\ )}{\partial t}+\vec{V}\cdot\nabla(\ )$$

< 2.9. Substantial derivative >

❖ Substantial derivative definition

- The rightmost compact D/Dt definition contains two terms.

  - The first $\partial/\partial t$ term is called *<u>the local derivative</u>*.

  - The second $\boldsymbol{V}\cdot\nabla$ term is called *<u>the convective derivative</u>*.

  - In steady flows, $\partial/\partial t = 0$, and only the convective derivative contributes.

< 2.10. Fundamental equations >

❖ Recast governing equations

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

< 2.10. Fundamental equations >

❖ Recast governing equations

- Momentum equation

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot \left(\rho u \vec{V}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \left(F_x\right)_{viscous}$$

$$u\frac{\partial \rho}{\partial t} + u\nabla \cdot \left(\rho \vec{V}\right) + \rho\frac{\partial u}{\partial t} + \rho \vec{V} \cdot \nabla u = -\frac{\partial p}{\partial x} + \rho g_x + \left(F_x\right)_{viscous}$$

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \left(F_x\right)_{viscous}$$

- In the same manner,

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y + \left(F_y\right)_{viscous}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + \left(F_z\right)_{viscous}$$

< 2.10. Fundamental equations >

❖ Recast governing equations

- Energy equation

$$\rho \frac{D\left(e + V^2/2\right)}{Dt} = \rho\dot{q} - \nabla \cdot \left(p\vec{V}\right) + \rho\left(\vec{f} \cdot \vec{V}\right) + \dot{Q}'_{viscous} + \dot{W}'_{viscous}$$

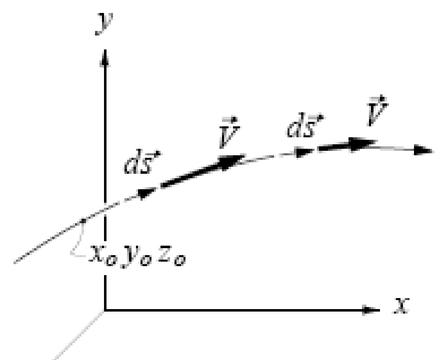
< 2.11. Pathline, Streamline, and Streakline >

❖ Streamline

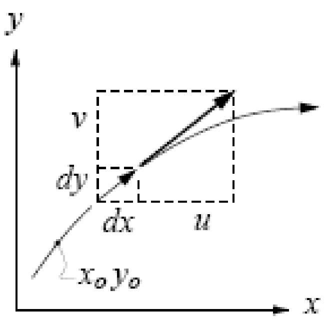
- A *streamline* is defined as a line which is everywhere parallel to the local velocity vector ***V****(x,y,z,t)*= *u****i****+v****j****+w****k***. Define

$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

as an infinitesimal arc-length vector along the streamline.



3-D streamline



2-D streamline

< 2.11. Pathline, Streamline, and Streakline >

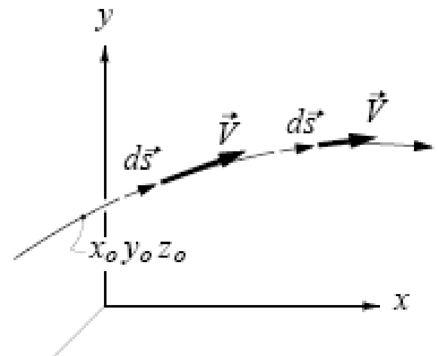
❖ Streamline

- Since this is parallel to ***V***, we must have

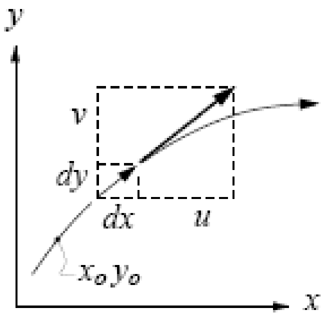
$$d\vec{s} \times \vec{V} = 0$$

$$(wdy - vdz)\hat{i} + (udz - wdx)\hat{j} + (vdx - udy)\hat{k} = 0$$

- Separately setting each component to zero gives three differential equations which define the streamline.



3−D streamline



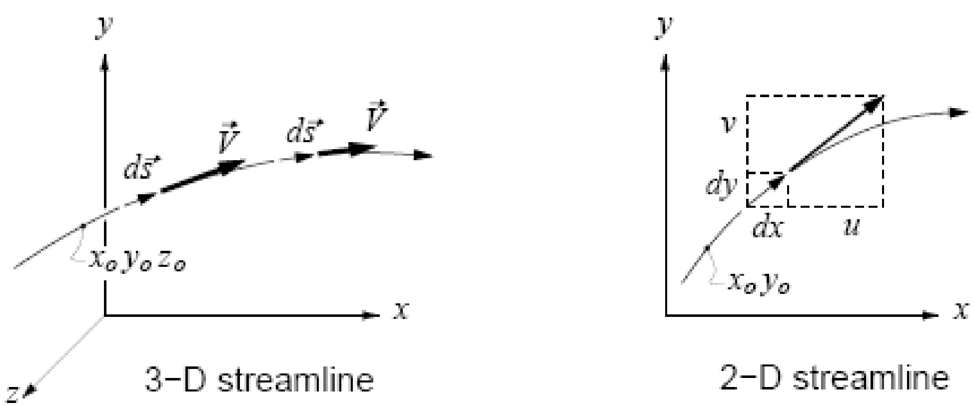
2−D streamline

< 2.11. Pathline, Streamline, and Streakline >

Streamline

- In 2-D we have $dz=0$ and $w=0$, and only the $\boldsymbol{k}$ component of the equation above is non-trivial. It can be written as an Ordinary Differential Equation for the streamline shape $y(x)$.

$$\frac{dy}{dx} = \frac{v}{u}$$



3−D streamline　　2−D streamline

< 2.11. Pathline, Streamline, and Streakline >

❖ Pathline

- The *pathline* of a fluid element A is simply the path it takes through space as a function of time.

- An example of a pathline is the trajectory taken by one puff of smoke which is carried by the steady or unsteady wind.

< 2.11. Pathline, Streamline, and Streakline >

❖ Streakline

- A *streakline* is associated with a particular point P in space which has the fluid moving past it.

- An example of a streakline is the continuous line of smoke emitted by a chimney at point P, which will have some curved shape if the wind has a time-varying direction.

< 2.11. Pathline, Streamline, and Streakline >

❖ Streakline

- Unlike a pathline, which involves the motion of only one fluid element A in time, a streakline involves the motion of all the fluid elements along its length.

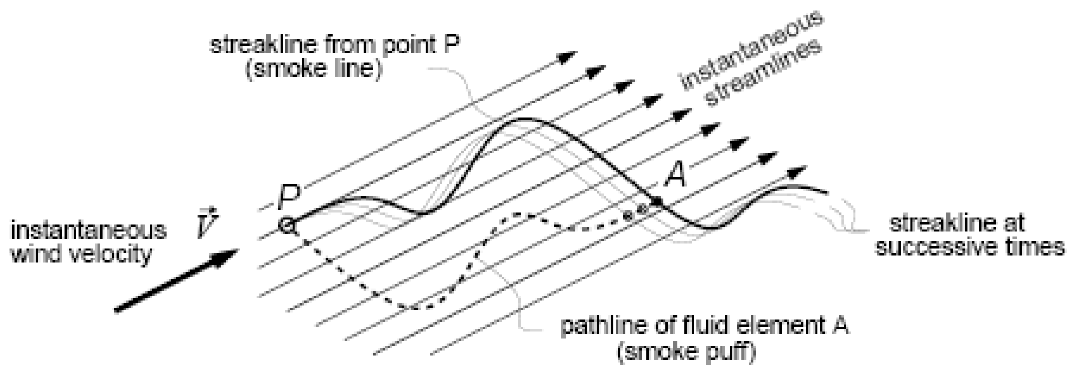
- Hence, the trajectory equations for a pathline are applied to all the fluid elements defining the streakline.

< 2.11. Pathline, Streamline, and Streakline >

❖ Summary

- The figure below illustrates streamlines, pathlines, and streaklines for the case of a smoke being continuously emitted by a chimney at point P, in the presence of a shifting wind. One particular smoke puff A is also identified.

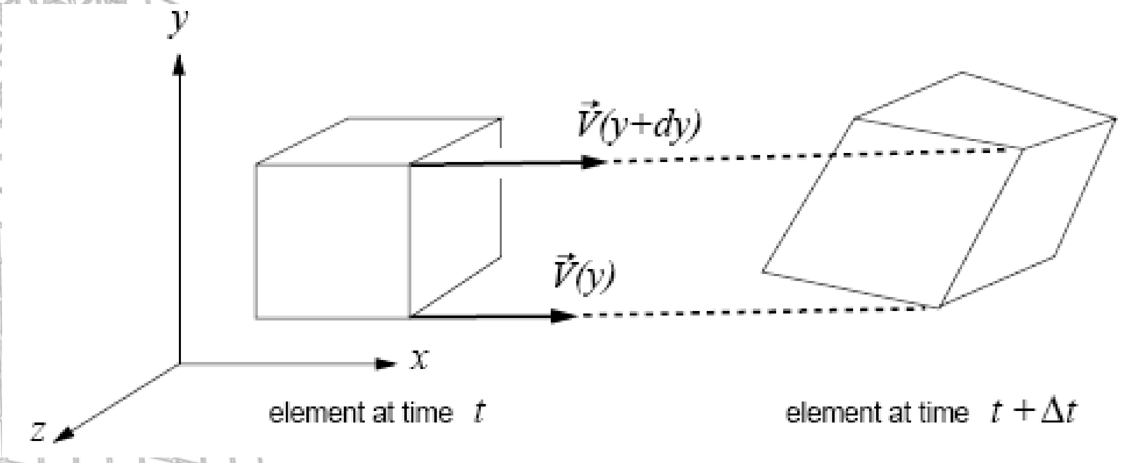
- In a *steady flow*, streamlines, pathlines, and streaklines *<u>all coincide</u>*.



< 2.12. Angular velocity, Vorticity, and Strain >

❖ Fluid element behavior

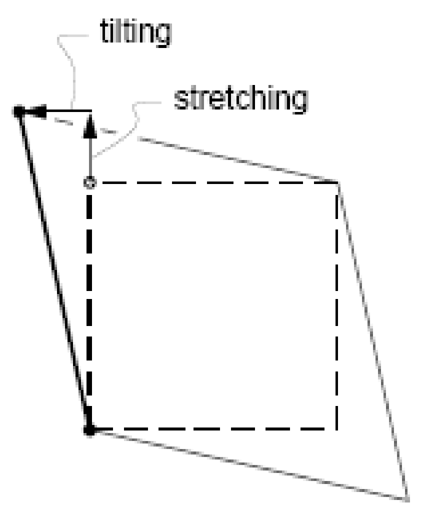
- Consider a moving fluid element which is initially rectangular, as shown in the figure. If the velocity varies significantly across the extent of the element, its corners will not move in unison, and the element will *rotate* and become *distorted*.



< 2.12. Angular velocity, Vorticity, and Strain >

❖ Fluid element behavior

- In general, the edges of the element can undergo some combination of *tilting* and *stretching*. For now we will consider only the tilting motions, because this has by far the greatest implications for aerodynamics.

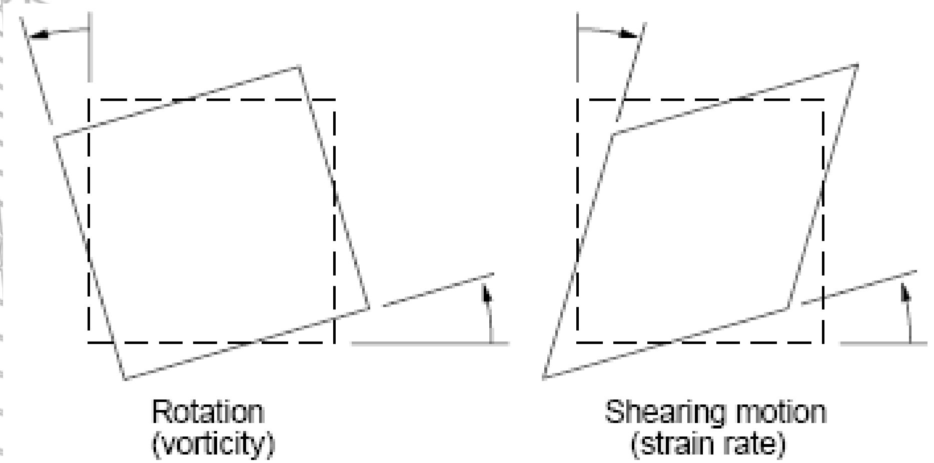


General edge movement

< 2.12. Angular velocity, Vorticity, and Strain >

❖ Fluid element behavior

- The figure below on the right shows two particular types of element-side tilting motions. If adjacent sides tilt equally and in the same direction, we have pure *rotation*. If the adjacent sides tilt equally and in opposite directions, we have pure *shearing* motion.

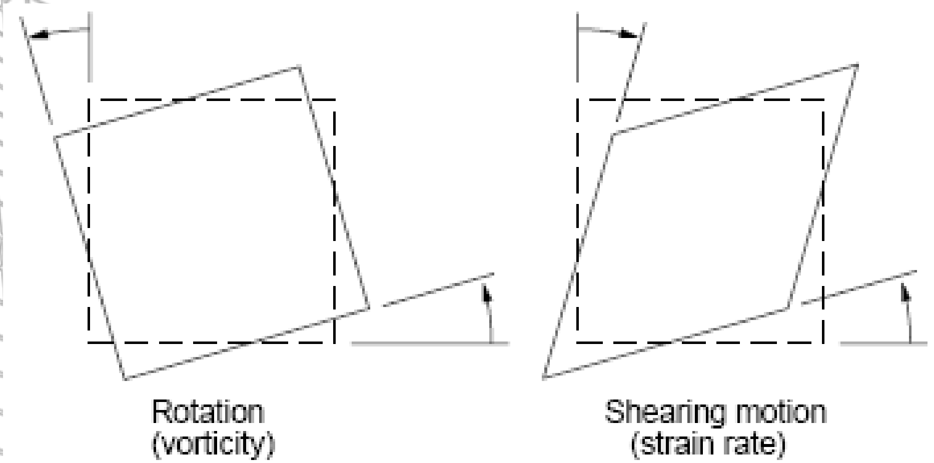


Tilting edge movements

< 2.12. Angular velocity, Vorticity, and Strain >

❖ Fluid element behavior

- Both of these motions have strong implications. *The absence of rotation will lead to a great simplification in the equations of fluid motion*. Shearing together with fluid viscosity produce shear stresses, which are responsible for phenomena like drag and flow separation.

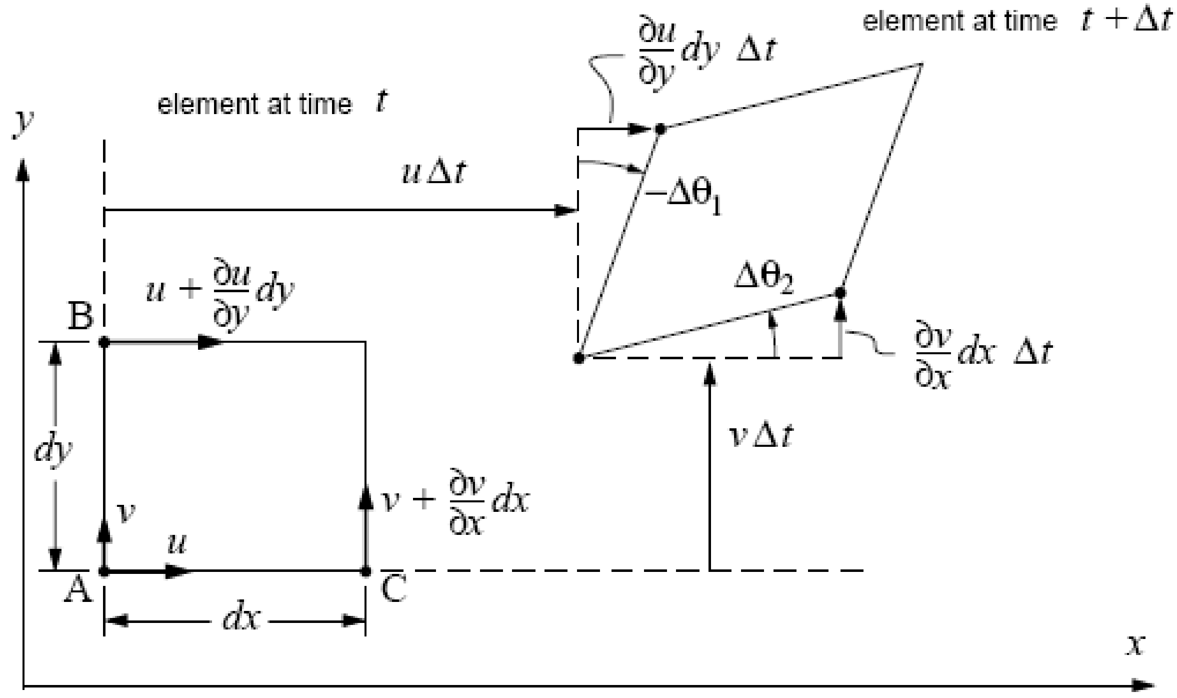


Tilting edge movements

< 2.12. Angular velocity, Vorticity, and Strain >

❖ Side tilting analysis

- Consider the 2-D element in the *xy* plane, at time *t*, and again at time *t+Δt*.

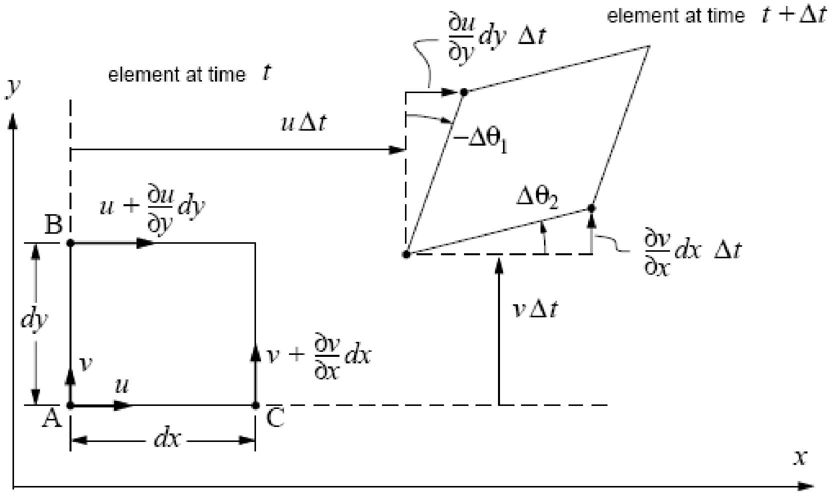


< 2.12. Angular velocity, Vorticity, and Strain >

❖ Side tilting analysis

- Points A and B have an x-velocity which differs by $\partial u/\partial y dy$. Over the time interval Δt they will then have a difference in x-displacements equal to

$$\Delta x_B - \Delta x_A = \frac{\partial u}{\partial y} dy \Delta t$$



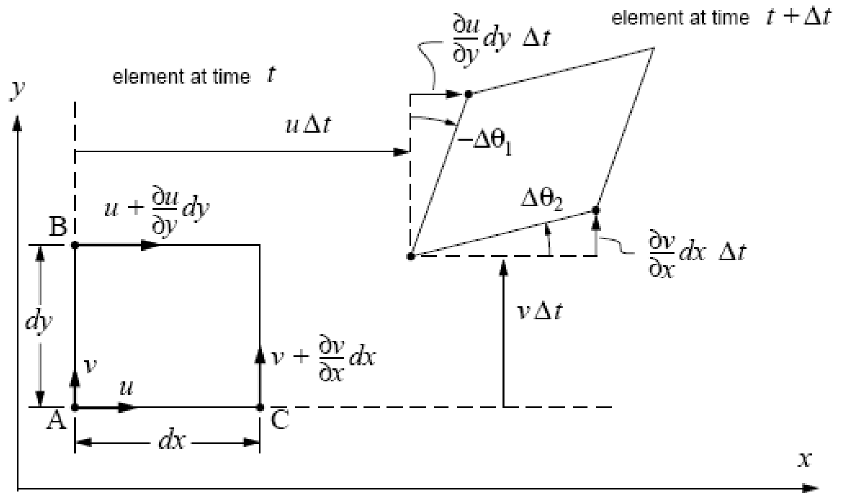
< 2.12. Angular velocity, Vorticity, and Strain >

❖ Side tilting analysis

- the associated angle change of side AB is

$$-\Delta\theta_1 = \frac{\Delta x_B - \Delta x_A}{dy} = \frac{\partial u}{\partial y}\Delta t$$

(assuming small angles)



< 2.12. Angular velocity, Vorticity, and Strain >

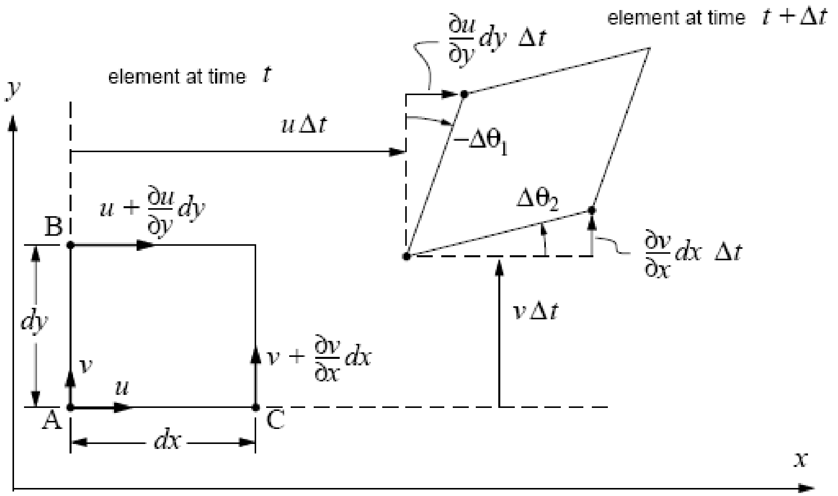
❖ Side tilting analysis

- A positive angle is defined counterclockwise. We now define a time rate of change of this angle as follows.

$$\frac{d\theta_1}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\theta_1}{\Delta t} = -\frac{\partial u}{\partial y}$$

- Similar analysis of the angle rate of side AC gives

$$\frac{d\theta_2}{dt} = \frac{\partial v}{\partial x}$$



< 2.12. Angular velocity, Vorticity, and Strain >

Angular velocity

- The angular velocity of the element, about the *z* axis in this case, is defined as the average angular velocity of sides AB and AC.

$$\omega_z = \frac{1}{2}\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

- The same analysis in the *xz* and *yz* planes will give a 3-D element's angular velocities ω_y and ω_x.

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

< 2.12. Angular velocity, Vorticity, and Strain >

❖ Angular velocity

- These three angular velocities are the components of the *angular velocity* vector

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\begin{pmatrix} \omega_x = \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right) \\ \omega_y = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right) \\ \omega_z = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right) \end{pmatrix}$$

< 2.12. Angular velocity, Vorticity, and Strain >

❖ Vorticity

- However, since 2$\boldsymbol{\omega}$ appears most frequently, it is convenient to define the *vorticity* vector $\boldsymbol{\xi}$ as simply twice $\boldsymbol{\omega}$.

$$\vec{\xi} = 2\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

- The components of the vorticity vector are recognized as the definitions of the curl of $\boldsymbol{V}$, hence we have

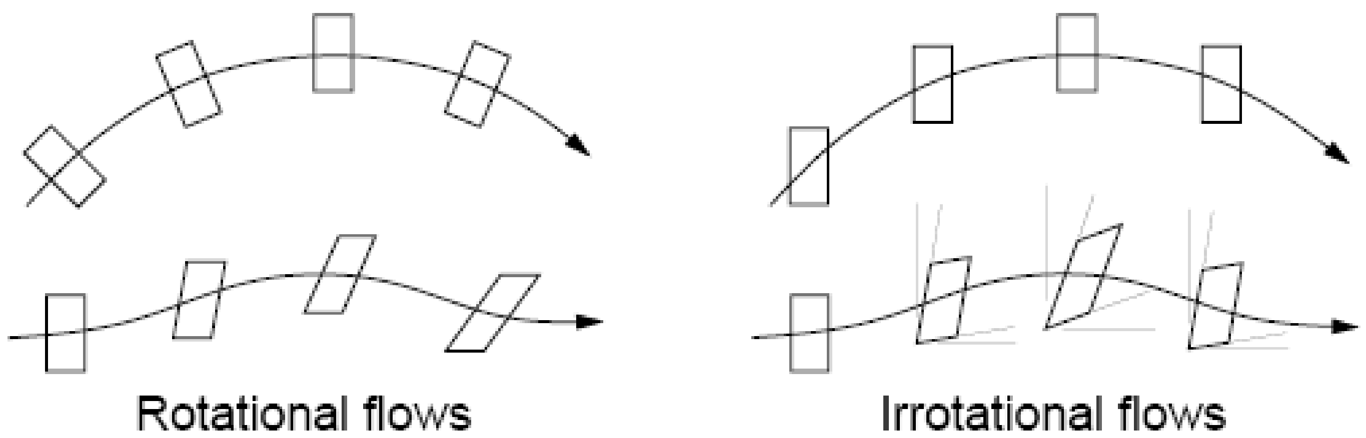
$$\vec{\xi} = \nabla \times \vec{V}$$

< 2.12. Angular velocity, Vorticity, and Strain >

❖ Vorticity

- Two types of flow can now be defined :

  - *Rotational flow*. Here $\nabla \times V \neq 0$ at every point in the flow. The fluid elements move and deform, and also rotate.

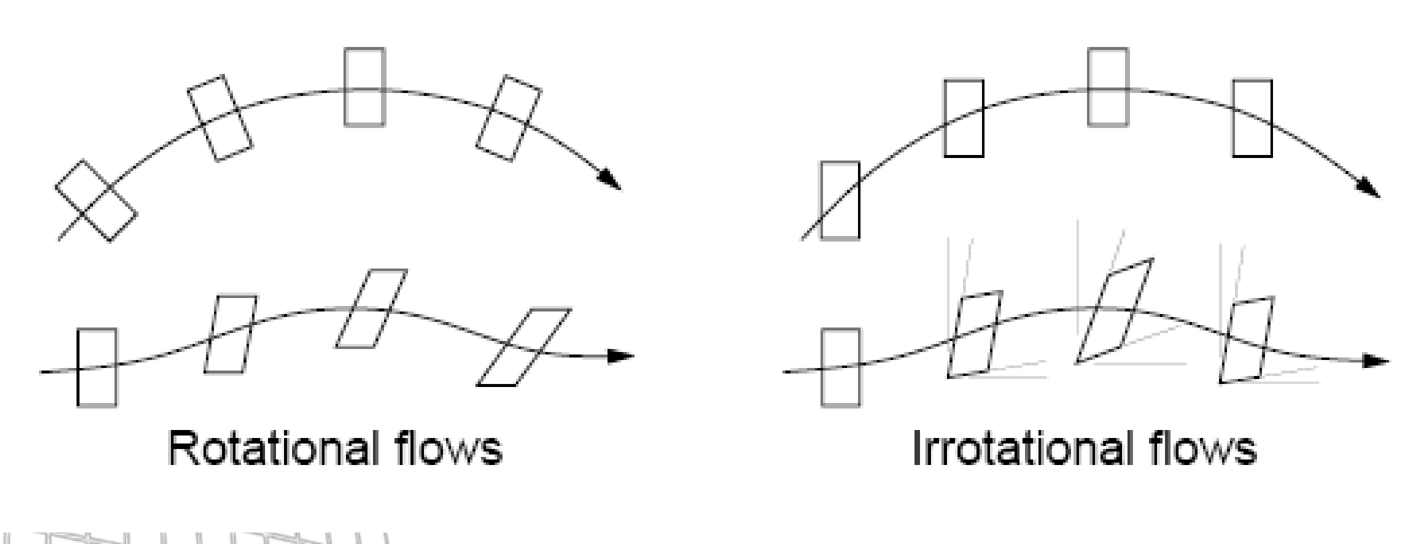


< 2.12. Angular velocity, Vorticity, and Strain >

❖ Vorticity

- Two types of flow can now be defined :

  - *Irrotational flow*. Here $\nabla \times V = 0$ at every point in the flow. The fluid elements move and deform, but do not rotate.

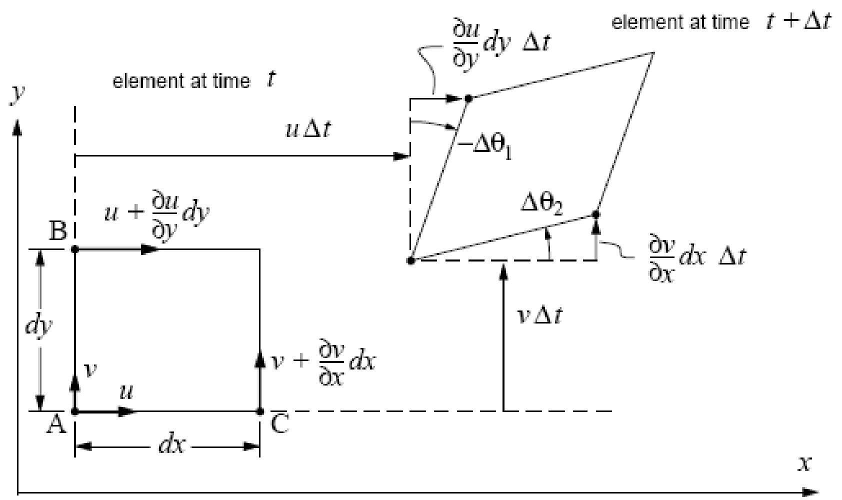


< 2.12. Angular velocity, Vorticity, and Strain >

❖ Strain rate

- Using the same element-side angles $\Delta\theta_1$, $\Delta\theta_2$, we can define the strain of the fluid element.

$$strain = \Delta\theta_2 - \Delta\theta_1$$

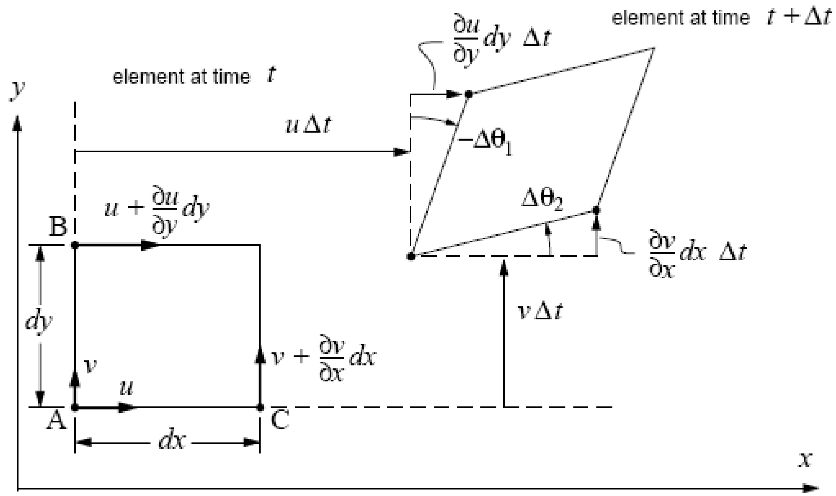


< 2.12. Angular velocity, Vorticity, and Strain >

❖ Strain rate

- This is the same as the strain used in solid mechanics. Here, we are more interested in the ***strain rate***, which is then simply

$$\frac{d(strain)}{dt} \equiv \varepsilon_{xy} = \frac{d\Delta\theta_2}{dt} - \frac{d\Delta\theta_1}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$



< 2.12. Angular velocity, Vorticity, and Strain >

❖ Strain rate

- Similarly, the strain rates in the *yz* and *zx* planes are

$$\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad \varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$